Optimal sensor placement for signal extraction

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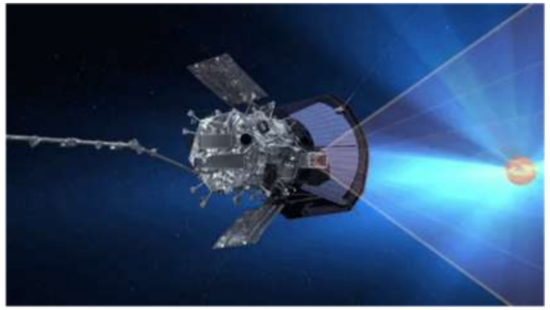
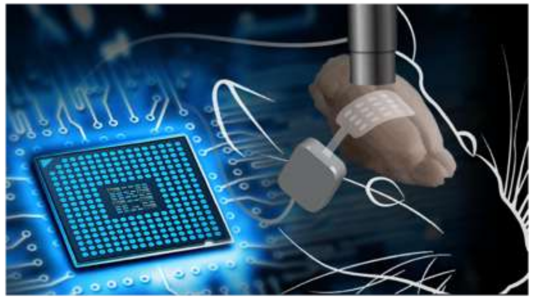
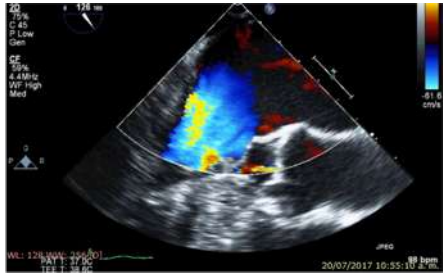
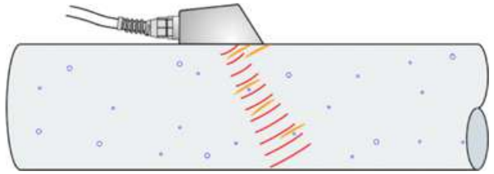
Sensors are being used in a variety of domains

- Industry
- Medicine
- Wireless communications
- Aerospace engineering
- Biomedical engineering
- Civil engineering
- Environmental study
- Robotics
- …



Optimal Sensor Placement

? Why optimal sensor placement is important?

✓ Limited number of sensors:

- **Economical**: reducing the price 
- **Energy**: reducing the required energy for the power supply 
- **Weight**: making the products as light as possible 
- **Computational complexity**: reducing computational cost 
- **Ergonomic design and arrangement** e.g. motion capture 

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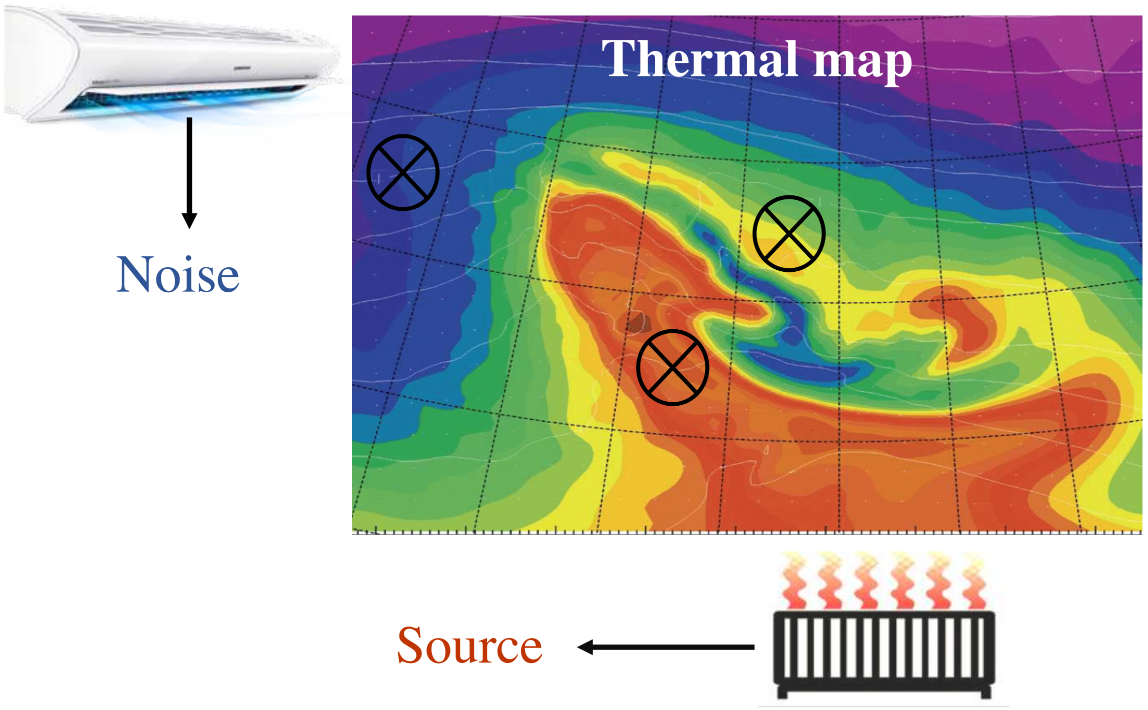
Optimal Sensor Placement for Source Extraction

Goal: extract the source S from a set of measurements



A set of measurements:

$$\mathbf{y}(\mathbf{X}_M, t) = \left\{ \begin{array}{c} y(\overset{\text{Location}}{\mathbf{x}_1}, \overset{\text{Time}}{t}) \\ y(\mathbf{x}_2, t) \\ \dots \\ y(\mathbf{x}_M, t) \end{array} \right\}$$

$$\mathbf{y}(\mathbf{X}_M, t) = \underbrace{\mathbf{a}(\mathbf{x}_M)}_{\text{Spatial gain}} \underbrace{s(t)}_{\text{Source}} + \underbrace{\mathbf{n}(\mathbf{x}_M, t)}_{\text{Noise}}$$

Question: Where to put the set of sensors to have the best source extraction?

Comparison with the previous works

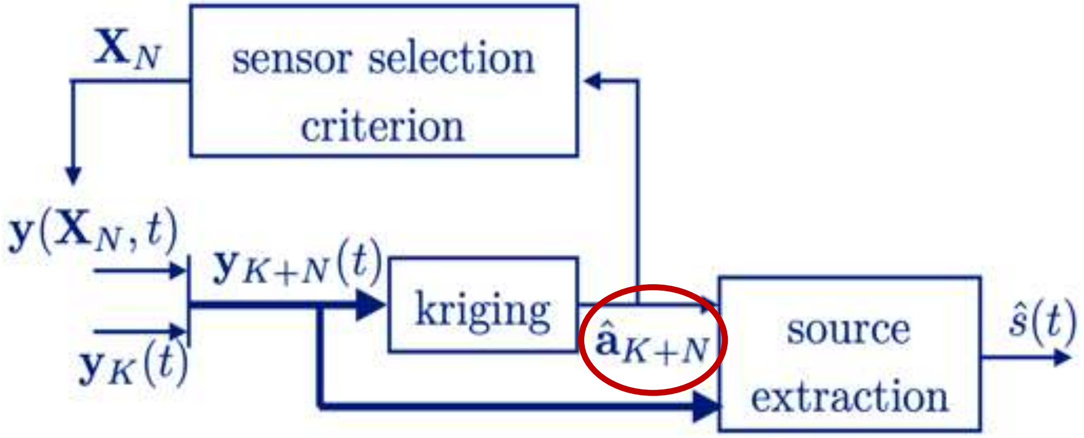
$$\mathbf{y}(\mathbf{X}_M, t) = \mathbf{a}(\mathbf{X}_M)s(t) + \mathbf{n}(\mathbf{X}_M, t)$$

Already selected positions

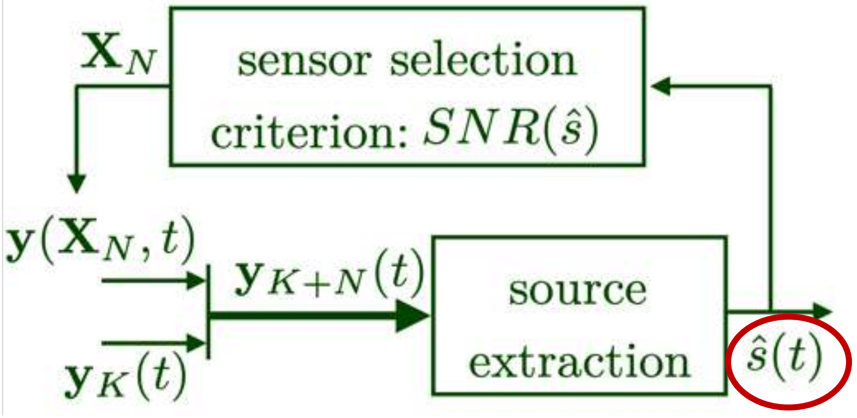
$$\mathbf{X}_M = \begin{Bmatrix} \mathbf{X}_K \\ \mathbf{X}_N \end{Bmatrix} \Longrightarrow \mathbf{y}_M = \begin{Bmatrix} \mathbf{y}_K \\ \mathbf{y}_N \end{Bmatrix}, \quad \mathbf{a}_M = \begin{Bmatrix} \mathbf{a}_K \\ \mathbf{a}_N \end{Bmatrix}$$

New positions

$$f_P = f(\mathbf{X}_P)$$



(a) Kriging approach

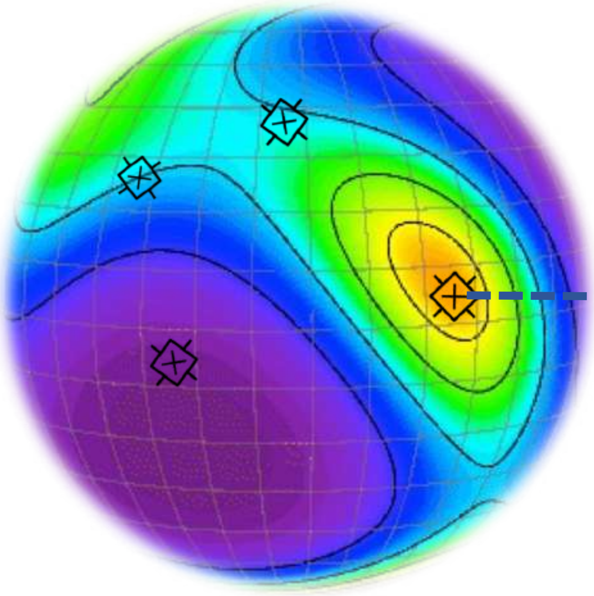


(b) Proposed approach

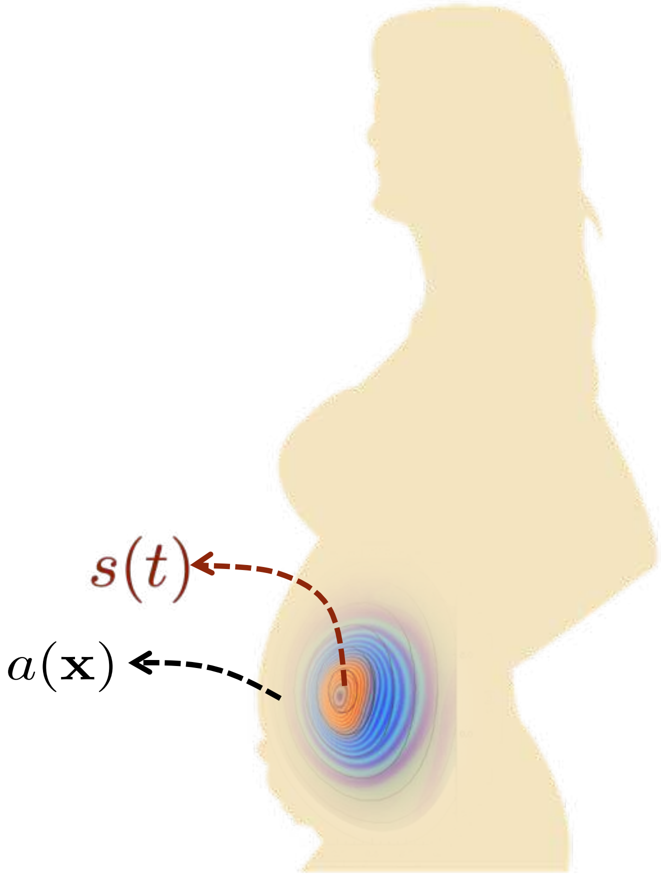
Optimal sensor placement for

Signal Extraction

Optimal sensor placement for Signal Extraction



$$\underbrace{\mathbf{y}(\mathbf{X}_M, t)}_{\text{Measurements}} = \underbrace{\mathbf{a}(\mathbf{X}_M)}_{\text{Spatial gain}} \underbrace{s(t)}_{\text{Source}} + \underbrace{\mathbf{n}(\mathbf{X}_M, t)}_{\text{Noise}}$$



$$\boxed{\mathbf{y}(\mathbf{X}_M, t) \xrightarrow{\;?\;} \hat{s}(t)}$$

Set of measurements

$$\mathbf{y}(\mathbf{X}_M, t) = \mathbf{a}(\mathbf{x}_M)s(t) + \mathbf{n}(\mathbf{x}_M, t) \xrightarrow{?} \hat{s}(t)$$

linear source extraction $\xrightarrow{\mathbf{f}\text{ : extraction vector}}$ $\hat{s}(t) = \mathbf{f}^T\mathbf{y}(\mathbf{X}_M, t)$ $\mathbf{f}$?

$$\hat{s}(t) = \underbrace{\mathbf{f}^T\mathbf{a}(\mathbf{X}_M)s(t)} + \underbrace{\mathbf{f}^T\mathbf{n}(\mathbf{X}_M, t)} \dashrightarrow SNR(\mathbf{f}) = \frac{\mathbb{E}\left[\left(\mathbf{f}^T\mathbf{a}_M s(t)\right)^2\right]}{\mathbb{E}\left[\left(\mathbf{f}^T\mathbf{n}_M(t)\right)^2\right]}$$

Criterion:
f with the best SNR

$$\boxed{SNR(\mathbf{f}) = \frac{\sigma_s^2\,\mathbf{f}^T\mathbf{a}_M\mathbf{a}_M^T\mathbf{f}}{\mathbf{f}^T R_M^n\mathbf{f}}}$$

$$\begin{cases} \sigma_S^2 = \mathbb{E}[s(t)^2] & \text{: Variance of the source (temporal variance)} \\ R_M^n = \mathbb{E}[\mathbf{n}_M(t)\mathbf{n}_M^T(t)] & \text{: Variance of the noise (spatial variance)} \end{cases}$$

Optimal sensor placement for Signal Extraction

$$\mathbf{y}(\mathbf{X}_M, t) = \mathbf{a}(\mathbf{x}_M)s(t) + \mathbf{n}(\mathbf{x}_M, t) \xrightarrow{?} \hat{s}(t)$$

linear source extraction $\xrightarrow{\mathbf{f}\text{ : extraction vector}}$ $\hat{s}(t) = \mathbf{f}^T \mathbf{y}(\mathbf{X}_M, t)$ $\mathbf{f}$?

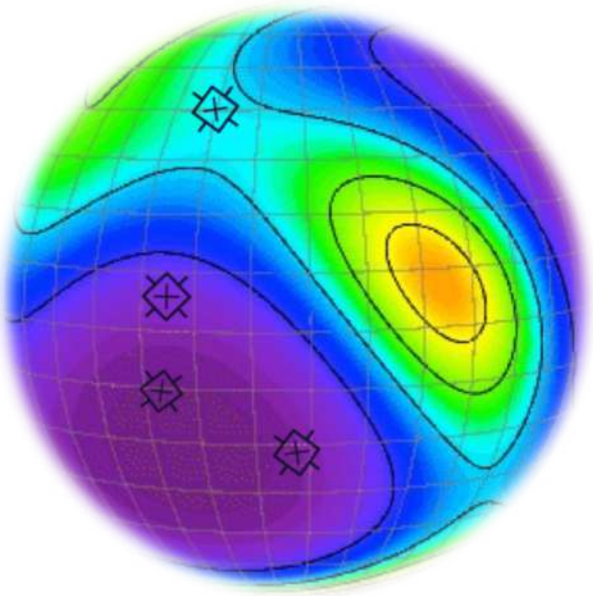
$$SNR(\mathbf{f}) = \frac{\sigma_s^2 \, \mathbf{f}^T \mathbf{a}_M \mathbf{a}_M^T \mathbf{f}}{\mathbf{f}^T R_M^n \mathbf{f}}$$

$$\xrightarrow{\text{Max}\{\text{SNR}\}} \begin{cases} \mathbf{f}^* = (R_M^n)^{-1} \mathbf{a}_M \\ \boxed{SNR(\mathbf{f}^*) = \sigma_S^2 \, \mathbf{a}_M^T (R_M^n)^{-1} \mathbf{a}_M} \end{cases}$$

Optimal sensor placement

for Signal Extraction

Optimal sensor placement for signal extraction



Goal: to have the best estimation of the source S

$$SNR(\mathbf{f}^*) = \sigma_S^2\, \mathbf{a}_M^T (R_M^n)^{-1} \mathbf{a}_M = J(\mathbf{X}_M)$$

$$\mathbf{X}_M^* = \arg\max_{\mathbf{X}_M} J(\mathbf{X}_M)$$

Combinatorial search ⟹ **high computational cost**

✓ **Greedy approach:**

selects the M sensors by sequentially selecting $N < M$

Optimal sensor placement for signal extraction

✓ **Greedy approach:**

Assumption:

- K sensors are already allocated
- Choosing other N sensor locations
  - $\mathbf{X}_M$ is the union of $\mathbf{X}_K$ and $\mathbf{X}_N$: $\mathbf{X}_M = \left\{ \begin{array}{c} \mathbf{X}_K \\ \mathbf{X}_N \end{array} \right\}$

$$\text{SNR:}\quad J(\mathbf{X}_M) = \mathbf{a}_M^T (R_M^n)^{-1} \mathbf{a}_M \longrightarrow J(\mathbf{X}_N|\mathbf{X}_K) = [\mathbf{a}_{K+N}^T (R_{K+N}^n)^{-1} \mathbf{a}_{K+N} | X_K]$$

$\hat{\mathbf{a}}_K$ and $R_K^n \longrightarrow$ independent of $\mathbf{X}_N$

$\hat{\mathbf{x}}_M$ ✓ $\dashrightarrow$ $\hat{\mathbf{f}}_M = (R_M^n)^{-1} \mathbf{m}_M^a$ ✓ $\dashrightarrow$ $\hat{s}(t)$

Optimal sensor placement for signal extraction

$$J(\mathbf{X}_N|\mathbf{X}_K) = [\mathbf{a}_{K+N}^T (R_{K+N}^n)^{-1} \mathbf{a}_{K+N} | X_K]$$

$a(\mathbf{x})$ ------> Imperfectly known : $\hat{a}(\mathbf{x})$

Modeling assumption:

✓ **stochastic Gaussian process** ------> $\hat{a}(\mathbf{x}) \sim \mathcal{GP}(m^a(\mathbf{x}), R^a(\mathbf{x}, \mathbf{x}'))$

Prior knowledge

Uncertainty

$$\hat{J}(\mathbf{X}_N|\mathbf{X}_K) = \mathbb{E}[\mathbf{a}_{K+N}^T (R_{K+N}^n)^{-1} \mathbf{a}_{K+N} | X_K] = (\mathbf{m}_{K+N}^a)^T (R_{K+N}^n)^{-1} \mathbf{m}_{K+N}^a + \mathrm{Tr}[(R_{K+N}^n)^{-1} R_{K+N}^a]$$

$$\hat{\mathbf{X}}_N = \arg\max_{\mathbf{X}_N} \hat{J}(\mathbf{X}_N|X_K)$$

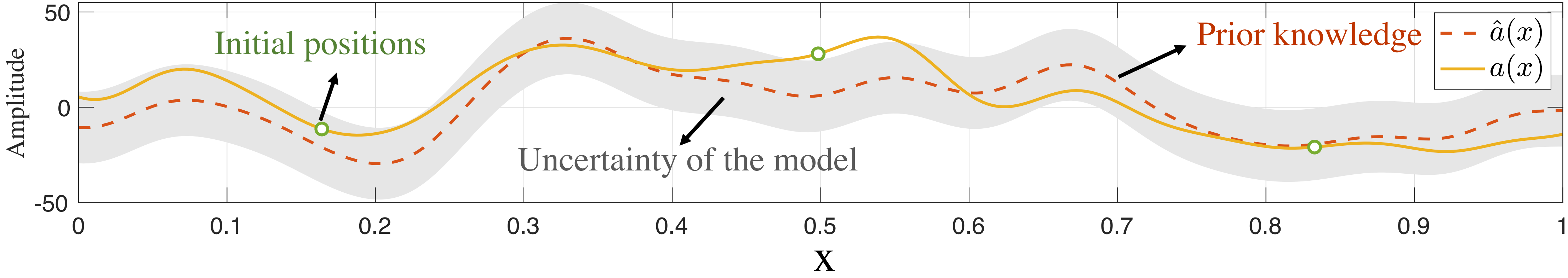
Numerical Results

Numerical Setup

- Synthetically generated data
- D (space dimension): 1
- x : normalized in the range $x \in [0, 1]$
  - a grid of size 200
  - three initial sensors at $X_K = \{0.165, 0.5, 0.835\}$
- $a(x)$ and $n(x)$: produced from Gaussian processes $\mathcal{GP}(m(x), C(x, x'))$
  - square exponential covariance function $C(x, x') = \sigma^2 \exp(-(x - x')^2/(2\rho^2))$
  - $m^n(x) = 0$
  - $m^a(x)$: randomly generated by a GP with a zero-mean and $\rho = 0.2$
  - ρ_a and ρ_n : taking different values

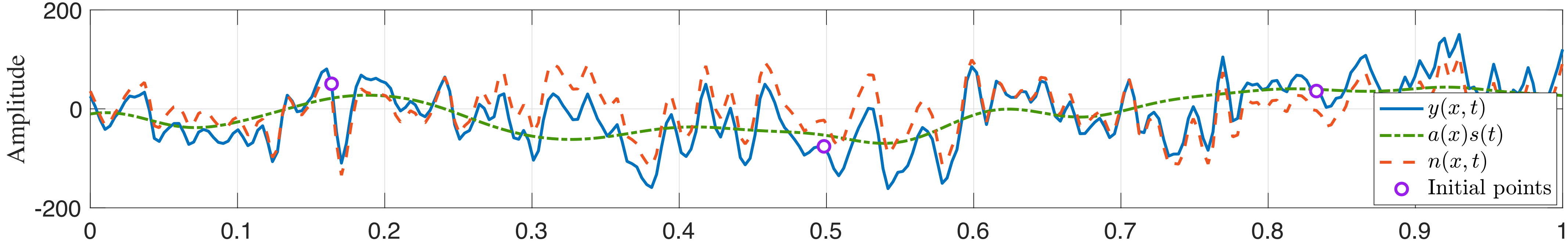
ρ : smoothness parameter

$\rho_a = 0.05$



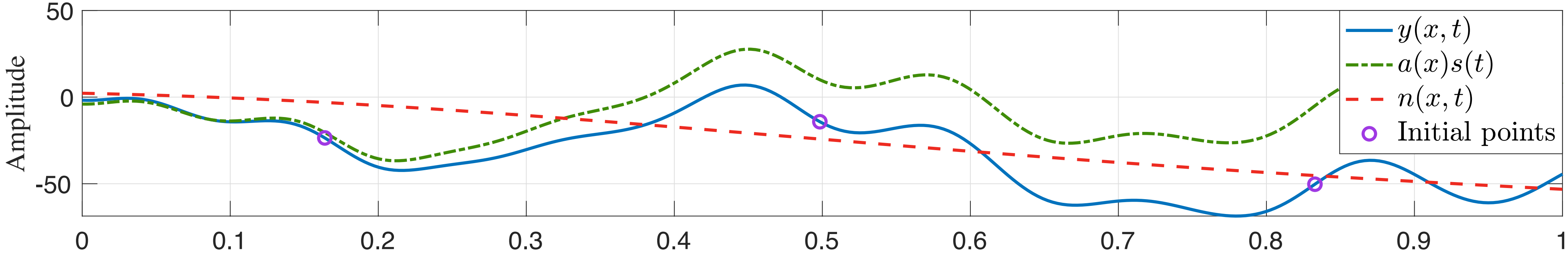
$\rho_n = 0.005$

(~ White noise)

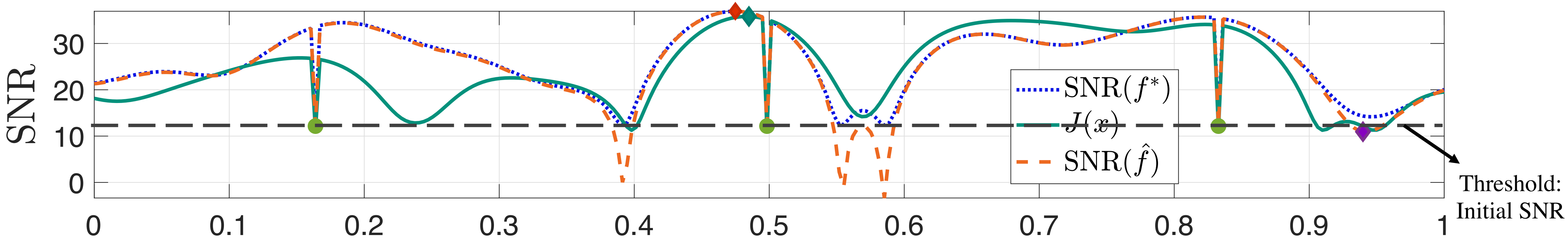


$\rho_n = 0.5$

(~ Smooth noise)



Numerical Results



The mean of SNR with uncertainty on 'a' : $\hat{J}(\mathbf{X}_N|\mathbf{X}_K) = \mathbb{E}[\hat{\mathbf{a}}_{K+N}^T (R_{K+N}^n)^{-1} \hat{\mathbf{a}}_{K+N} \,|\, \mathbf{X}_K]$

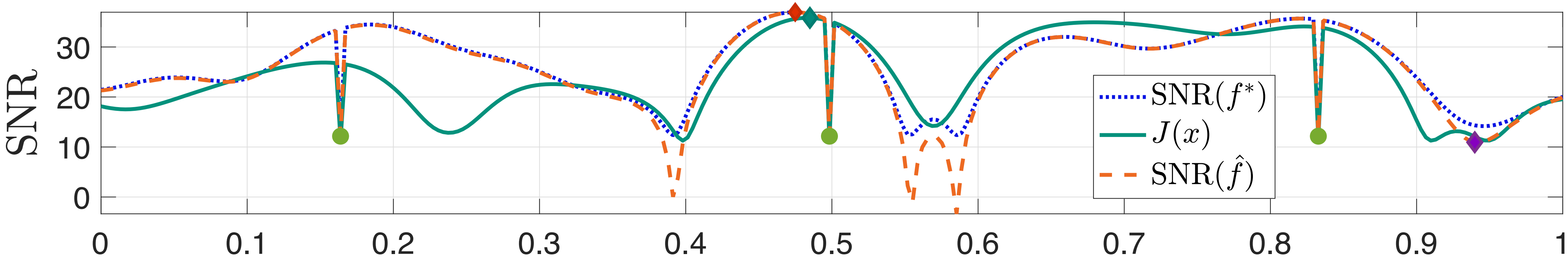
The true SNR with the true values of 'a' : $SNR(\mathbf{f}^*) = \sigma_S^2\, \mathbf{a}_M^T (R_M^n)^{-1} \mathbf{a}_M$

The true SNR with uncertainty on 'a' : $\hat{\mathbf{f}}_M = (R_M^n)^{-1} \mathbf{m}_M^a$ ----> $SNR(\mathbf{f}) = \dfrac{\mathbb{E}\left[\left(\mathbf{f}^T \mathbf{a}_M s(t)\right)^2\right]}{\mathbb{E}\left[\left(\mathbf{f}^T \mathbf{n}_M(t)\right)^2\right]}$

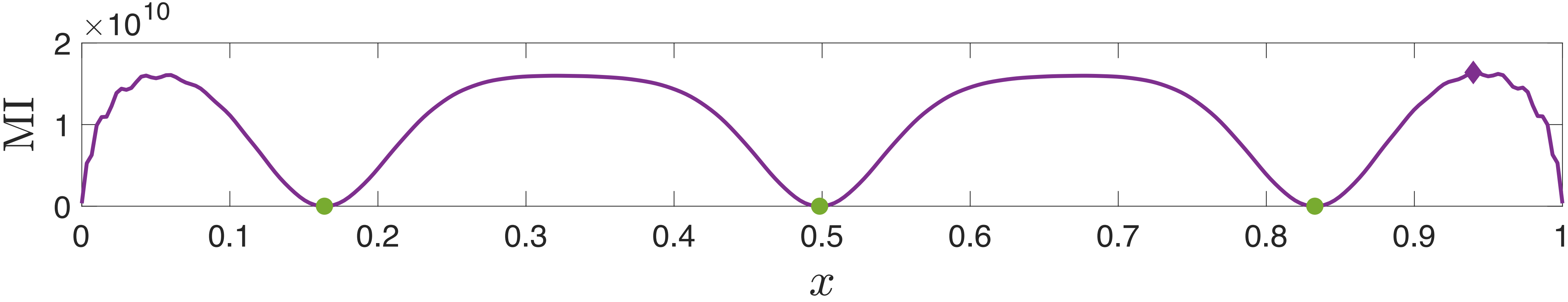
$SNR(\mathbf{X}_N) < SNR(\mathbf{X}_K) \Longrightarrow$ Failure

Numerical Results

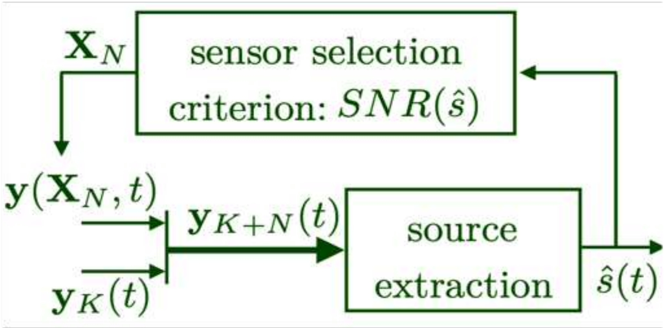
Proposed method:



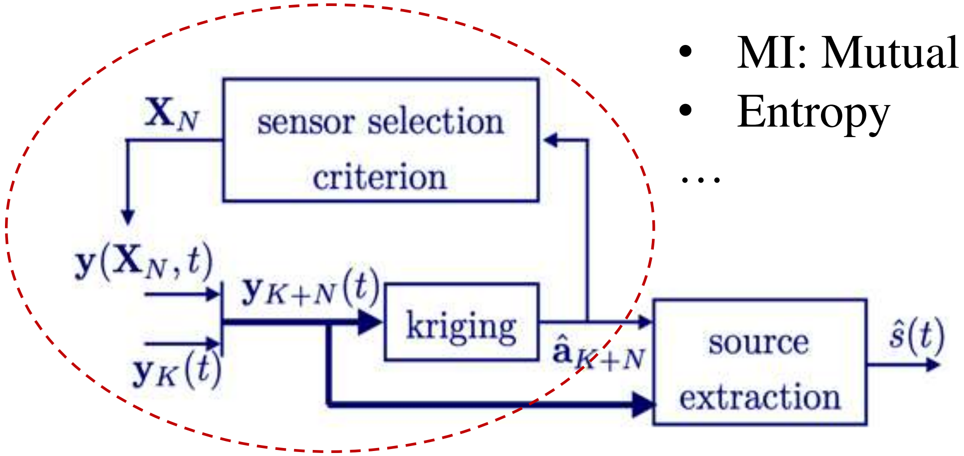
Classical Kriging:



Numerical Results



(a) Proposed approach



(b) Kriging approach

- MI: Mutual information
- Entropy

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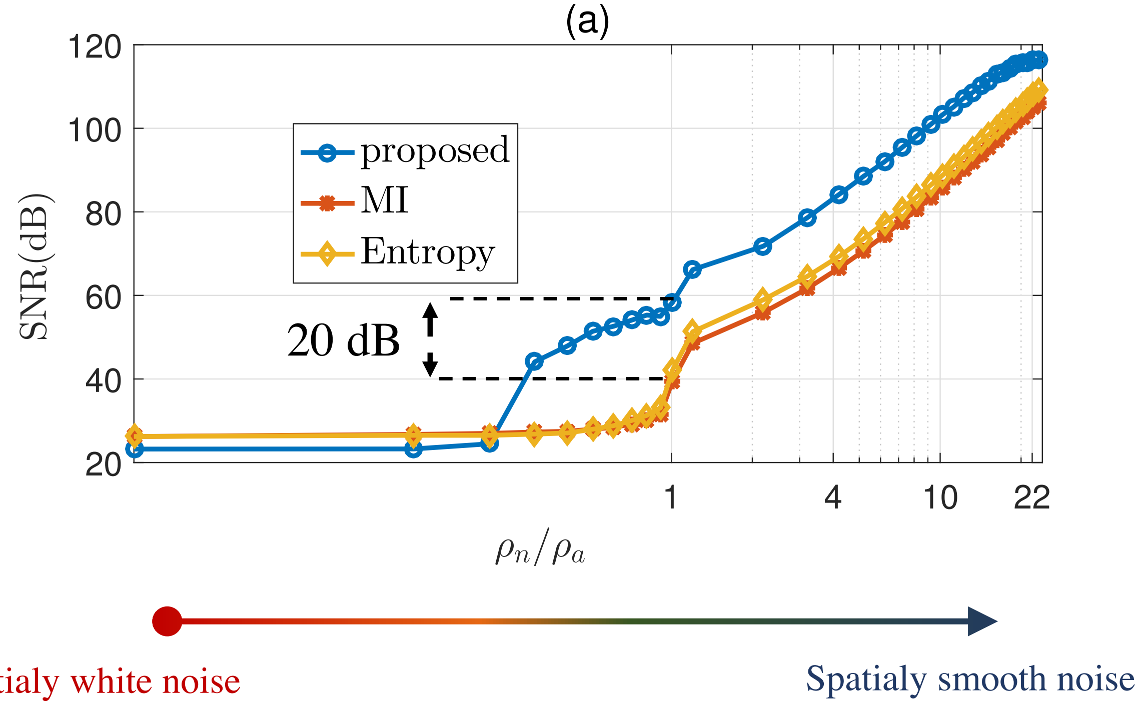
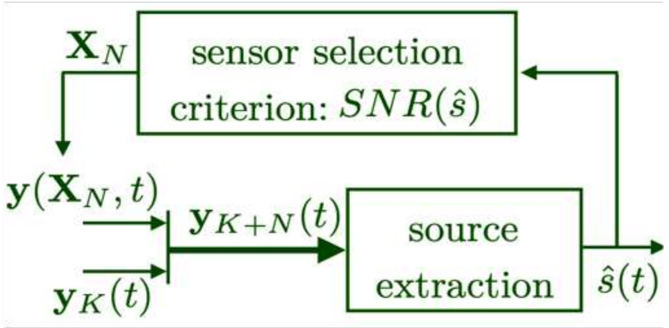
$\rho_a = 0.05$

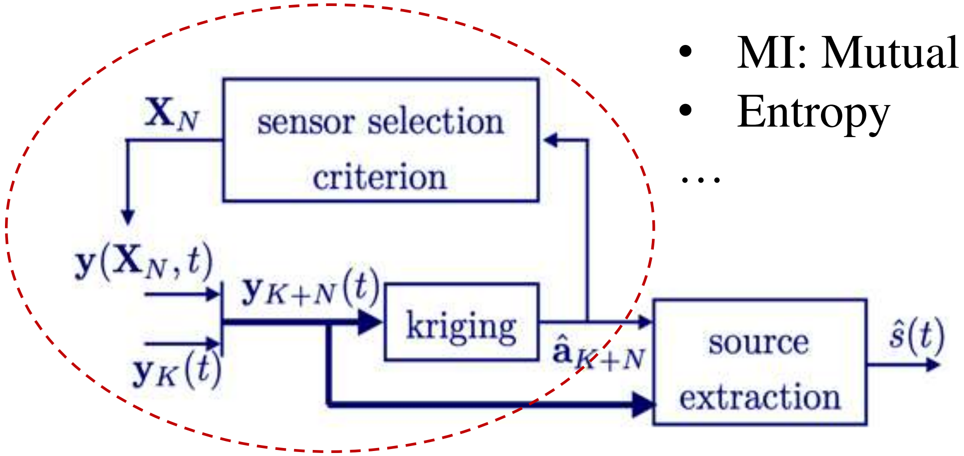
50 data sets were randomly generated



Numerical Results



(a) Proposed approach



(b) Kriging approach

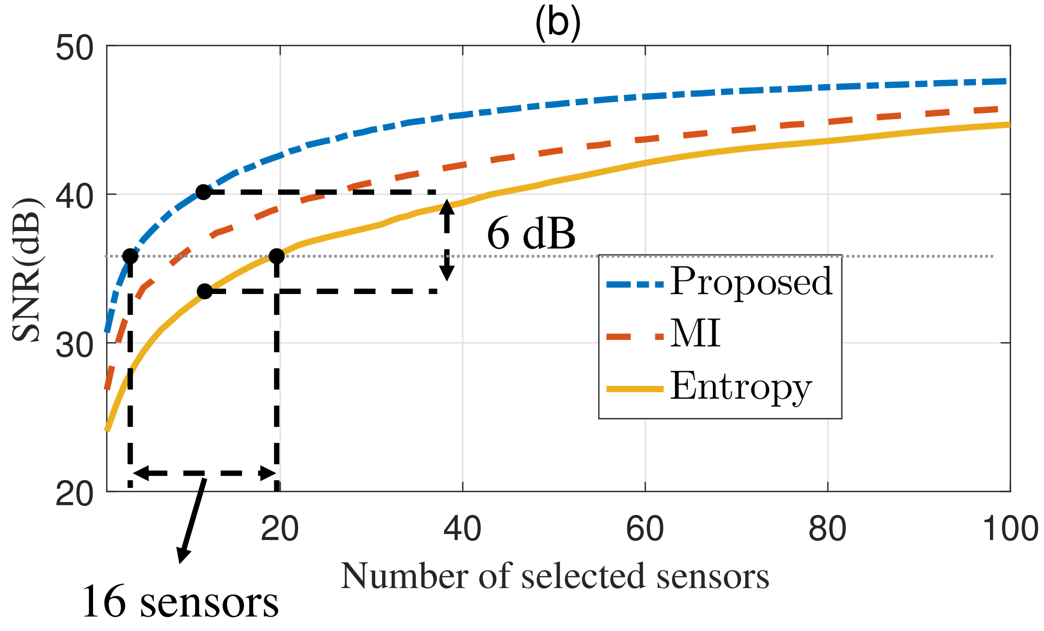
- MI: Mutual information
- Entropy

…

$\rho_a = 0.05$

$\rho_n / \rho_a = 0.01$

50 data sets were randomly generated



Conclusions & Perspectives

Conclusions

- The problem of optimal sensor placement:
  - signal extraction
  - using noisy underdetermined measurements
  - limited number of sensors
- Criterion to select sensor locations:
  - The average output SNR
    - SNR of the linearly extracted signal
  - considering the uncertainty on the spatial gain
  - greedy method
- Numerical simulations:
  - superior efficiency and accuracy of the proposed method in the source extraction problem compared to classical kriging methods

Perspectives

- Analytic characterization of the regions where placing new sensors deteriorate the extraction (failure study)
- Extension: considering an imperfectly known noise covariance kernel
- Model-based approach, requiring knowledge on the spatial gains and noise:
  - Tackling the optimal sensor placement problem <u>in a data driven approach</u>, e.g. using independent component analysis (ICA).
- Adjustment by adding new sensors

Thank you!